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Unified Krylov–Bogoliubov–Mitropolskii method for solving *n*th order non-linear systems with slowly varying coefficients

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Abstract

The unified Krylov–Bogoliubov-Mitropolskii (KBM) method is extended for obtaining the transient response of an n-th order ($n \geqslant 2$) non-linear system with slowly varying coefficients. The method is a generalization of KBM method and covers all the three cases when the eigenvalues of the unperturbed equation are real, complex conjugate, or purely imaginary. It is shown that by suitable substitution for the eigenvalues in the general result that the solution corresponding to each of the three cases can be obtained. The method is illustrated by examples.

1. Introduction

By means of an extension of the unified Krylov–Bogoliubov–Mitropolskii (KBM) [1–5] method, the perturbation solution of an nth order ($n \geqslant 2$) weakly non-linear system with slowly varying coefficients is found. The method is a generalization of extended (by Popov [4]) KBM [1–3] method and covers under-damped and over-damped systems. Shamsul Alam [6] extended the unified method to the critically damped non-linear system. Bojadziev and Edward [7] studied some under-damped and over-damped systems with slowly varying coefficients.

Some authors extended this method to higher order non-linear systems. Mulholland [8], Osiniskii [9] and Bojadziev [10] investigated third order oscillations. Sattar [11] studied a third order over-damped system. Shamsul and Sattar [12,13] extended the unified method to third-order systems. An n-dimensional biological system was studied by Pavilidis [14]. Recently, Shamsul Alam [15] has extended the unified method to an nth order non-linear system.

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In this paper, the unified KBM method is further extended to an nth order non-linear system with slowly varying coefficients. The method covers under-damped (both small and significant damping) and over-damped systems. Thus the method is independent of whether the unperturbed system has n eigenvalues real, or complex conjugate, or pure imaginary. The method is also independent of the order of the system. The approach of this method is simple and a changed form of standard KBM method. However, the new solution can be brought to formal KBM solution by suitable substitutions.

2. The method

Consider a weakly non-linear system governed by an nth order differential equation

$$x^{(n)} + c_1(\tau)x^{(n-1)} + \cdots + c_n(\tau)x = \varepsilon f(x, \dot{x}, \ldots, x^{(n-1)}, \tau), \tag{1}$$

where $x^{(i)}, i = n, n-1, \ldots$ represents ith derivative, ε a small parameter, $\tau = \varepsilon t$ slowly varying time, $c_j(\tau) \geqslant 0$, $j = 1, 2, \ldots, n$ and f a non-linear function. The coefficients in Eq. (1) are slowly varying in that their time derivatives are proportional to ε [3].

Setting $\varepsilon = 0$, $\tau = \tau_0 = Const.$ in Eq. (1), we obtain the unperturbed solution of the equation. Let Eq. (1) has n eigenvalues $\lambda_j(\tau_0), j = 1, 2, \ldots, n$, where $\lambda_j(\tau_0)$ are constant, but when $\varepsilon \neq 0$, $\lambda_j(\tau)$ slowly vary with time. The unperturbed solution of Eq. (1) becomes

$$x(t, 0) = \sum_{j=1}^{n} a_{j,0} e^{\lambda_j(\tau_0)t}, \tag{2}$$

where $a_{j,0}$, $j = 1, 2 \ldots, n$ are arbitrary constants.

Now we seek a solution of Eq. (1) that reduces to Eq. (2) as a limit $\varepsilon \to 0$. Following the KBM method [1–3], we look for a solution

$$x(t, \varepsilon) = \sum_{j=1}^{n} a_j(t) + \varepsilon\, u_1(a_1, a_2, \ldots, a_n, \tau) + \varepsilon^2 u_2(a_1, a_2, \ldots, a_n, \tau) + \varepsilon^3 \cdots \tag{3}$$

in which each a_j satisfies a first order differential equation:

$$\dot{a}_j = -\lambda_j(\tau)\, a_j + \varepsilon A_j(a_1, a_2, \ldots, a_n, \tau) + \varepsilon^2 B_j(a_1, a_2, \ldots, a_n, \tau) + \varepsilon^3 \cdots. \tag{4}$$

Confining only to the first few terms, 1, 2, …, m, in the series expansions of Eqs. (3) and (4), we evaluate the functions u_1, u_2, … and A_j, B_j, …, $j = 1, 2, \ldots, n$ such that $a_j(t)$ appearing in Eqs. (3) and (4) satisfy the given differential equation (1) with an accuracy of ε^{m+1} [15]. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a low order, usually the first [5]. In order to determine these functions it is assumed that the functions u_1, u_2, … do not contain the fundamental terms [5–7,12] which are included in the series expansion (3) at order ε^0.

Differentiating $x(t,\varepsilon)$ n-times with respect to t, substituting derivatives $x^{(n)}, x^{(n-1)}, \ldots, \dot{x}$ and x in the original equation (1) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{n}(\Omega-\lambda_j)u_1+\sum_{j=1}^{n}\left(\prod_{k=1,k\neq j}^{n}(\Omega-\lambda_k)\right)A_j+\sum_{j=1}^{n}\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_j^{n-k-2}\right)\lambda_j' a_j$$
$$=f^{(0)}(a_1,a_2,\ldots,a_n,\tau), \tag{5}$$

where $\Omega=\sum_{j=1}^{n}\lambda_j a_j(\partial/\partial a_j)$, $\lambda_j'=\mathrm{d}\lambda_j/\mathrm{d}\tau,\ j=1,2,\ldots,n; f^{(0)}=f(x_0,\dot{x}_0,\ldots,x_0^{(n-1)})$, $x_0=\sum_{j=1}^{n}a_j(t)$.

We have already assumed that u_1 does not contain fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t, t\sin t$ and te^{-t} [2,10]. Under these restrictions, we are able to solve Eq. (5) by separating this into $n+1$ individual equations for the unknown functions u_1 and $A_j, j=1,2,\ldots,\ n$. In general, the functions $f^{(0)}$ and u_1 are expanded in Taylor's series

$$f^{(0)}=\sum_{m_1=0,m_2=0,\cdots,m_n=0}^{\infty,\infty,\cdots,\infty}F_{m_1,m_2,\cdots,m_n}(\tau)a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n} \tag{6}$$

and

$$u_1=\sum_{m_1=0,m_2=0,\ldots,m_n=0}^{\infty,\infty,\ldots,\infty}U_{m_1,m_2,\ldots,m_n}(\tau)a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}. \tag{7}$$

First of all, we may consider the situation when n is an even number and eigenvalues of the unperturbed equation are $-\mu_l(\tau_0)\pm\omega_l(\tau_0), l=1,2,\ldots,n/2$. For the above-imposed restrictions, it assures that u_1 must exclude all terms with $a_{2l-1}^{m_{2l-1}}\ a_{2l}^{m_{2l}}, l=1,2,\ldots,n/2$ of $f^{(0)}$ where $m_{2l-1}-m_{2l}=\pm1$, since as a linear approximation (i.e., $\varepsilon\to0$) $a_{2l-1}^{m_{2l-1}}a_{2l}^{m_{2l}}$ becomes $e^{\omega_l t}$ when $m_{2l-1}-m_{2l}=1$ or $e^{-\omega_l t}$ when $m_{2l-1}-m_{2l}=-1$. It is noted that $e^{\pm\omega_l t}, l=1,2,\ldots,n/2$ are known as fundamental terms [5–7,12]. Naturally these are included in equations of $A_j, j=1,2,\ldots,n/2$. Moreover, it is restricted (by Krylov et al. [1,2]) that the functions $A_j, j=1,2,\ldots,n$ are independent of fundamental terms. Hence for an even value of n, we obtain the following equations for u_1 and $A_j, j=1,2,\ldots,n$:

$$\prod_{j=1}^{n}(\Omega-\lambda_j)u_1=\sum_{m_1=0,m_2=0,\ldots,\ m_n=0}^{\infty,\infty,\ldots,\infty}F_{m_1,m_2,\ldots,m_n}(\tau)a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n},\ m_{2l-1}-m_{2l}\neq\pm1, \tag{8}$$

$$\left(\prod_{k=1,k\neq 2l-1}^{n}(\Omega-\lambda_k)\right)A_{2l-1}+\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l-1}^{2l-k-2}\right)\lambda_{2l-1}'a_{2l-1}$$
$$=\sum_{m_{2l-1}=0,m_{2l}=0}^{\infty,\infty}F_{0,0,\cdots,m_{2l-1},m_{2l},\ldots,0}a_{2l-1}^{m_{2l-1}}a_{2l}^{m_{2l}}, m_{2l-1}-m_{2l}=1, \tag{9}$$

and

$$\left(\prod_{k=1,k\neq 2l}^{n}(\Omega-\lambda_k)\right)A_{2l}+\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2}\right)\lambda'_{2l}a_{2l}$$
$$=\sum_{m_{2l-1}=0,m_{2l}=0}^{\infty,\infty}F_{0,0,\ldots,m_{2l-1},m_{2l},\ldots,0}a_{2l-1}^{m_{2l-1}}a_{2l}^{m_{2l}},\ m_{2l-1}-m_{2l}=-1. \quad (10)$$

It is very easy to determine the particular solutions of Eqs. (8)–(10). To do this we only replace operator Ω by $\sum_{j=1}^{n}m_j\lambda_j$, since we know $\Omega(a_1^{m_1}a_2^{m_2}\ldots a_n^{m_n})=\sum_{j=1}^{n}m_j\lambda_j(a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n})$. Thus the determination of first order solution (improved approximation) of (1) is clear for an even value of n. Now we shall consider the situation when n is odd. In this case, the eigenvalues of the unperturbed equation can be written as $-\mu_0(\tau_0)$, $-\mu_l(\tau_0)\pm\omega_l(\tau_0), l=1,2,\ldots,\ (n-1)/2$. It is clear that the above three equations (8)–(10) are still valid (here, the subscript of a, A and λ will be changed only, since we will start from a_2 instead of a_1) and one equation for A_1 will be added. To determine the equation of A_1, we shall follow the assumption of Bojadziev [10] that u_1 does contain a term te^{-t} (as limits $\mu_l\to 0$ for all l) and obtain the following equation:

$$\left(\prod_{k=2}^{n}(\Omega-\lambda_k)\right)A_1+\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_1^{2l-k-2}\right)\lambda'_1a_1$$
$$=\sum_{m_1,m_{2l}=0,m_{2l+1}=0}^{\infty,\infty}F_{m_1,0,\ldots,m_{2l},m_{2l+1},\ldots,0}\ a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},m_{2l}=m_{2l+1}. \quad (11)$$

Then the equations for u_1 and $A_j, j=2,3,\ \cdots,\ n$ are written as

$$\prod_{j=1}^{n}(\Omega-\lambda_j)u_1=\sum_{m_1=0,m_2=0,\ldots,m_n=0}^{\infty,\infty,\ldots,\infty}F_{m_1,m_2,\ldots,m_n}(\tau)a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n},\ m_{2l}-m_{2l+1}\neq 0,\pm 1, \quad (12)$$

$$\left(\prod_{k=1,k\neq 2l}^{n}(\Omega-\lambda_k)\right)A_{2l}+\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2}\right)\lambda'_{2l}a_{2l}$$
$$=\sum_{m_{2l}=0,m_{2l+1}=0}^{\infty,\infty}F_{0,0,\ldots,m_{2l},m_{2l+1},\ldots,0}\ a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ m_{2l}-m_{2l+1}=1, \quad (13)$$

and

$$\left(\prod_{k=1,k\neq 2l+1}^{n}(\Omega-\lambda_k)\right)A_{2l+1}+\tfrac{1}{2}\left(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l+1}^{2l-k-2}\right)\lambda'_{2l+1}a_{2l+1}$$
$$=\sum_{m_{2l}=0,m_{2l+1}=0}^{\infty,\infty}F_{0,0,\ldots,m_{2l},m_{2l+1},\ldots,0}\ a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ m_{2l}-m_{2l+1}=-1. \quad (14)$$

Eqs. (11)–(14) can be solved using a similar procedure discussed above, so that the determination of the first order solution of Eq. (1) is also clear for an odd value of n.

We have already mentioned that solution Eq. (3) is not a standard form of KBM method; we shall be able to transform (3) to the exact form of KBM solution by substitutions

$$a_{2l-1} = \tfrac{1}{2} b_l e^{\varphi_l},\\ a_{2l} = \pm \tfrac{1}{2} b_l e^{-\varphi_l}, \quad l = 1, 2, \ldots, n/2, \tag{15}$$

or

$$a_1 = b_0,\\ a_{2l} = \tfrac{1}{2} b_l e^{\varphi_l},\\ a_{2l+1} = \pm \tfrac{1}{2} b_l e^{-\varphi_l}, \quad l = 1, 2, \ldots, (n-1)/2. \tag{16}$$

3. Example

3.1. Second order nonlinear systems

To illustrate the method, we first consider the non-oscillation and oscillations of a pendulum with variable length. The differential equation of the motion is

$$\frac{d}{dt}\left(ml^2(\tau)\dot{x}\right) + 2\delta \frac{d}{dt}(l(\tau)x) + mgl(\tau) \sin x = 0, \tag{17}$$

where m is the mass, x the angle of deviation of the pendulum from the vertical, 2δ the coefficient of damping, g the acceleration of gravity, $l(\tau)$ the length of the pendulum varying slowly with time. For small oscillations we can use the first of the two terms of the development of $\sin x$. Then Eq. (17) can be written as

$$\ddot{x} + 2k(\tau)\dot{x} + v^2(\tau)x = -\varepsilon(\tau)\rho(\tau)[\dot{x} + k(\tau)x] + \varepsilon_1(\tau)x^3, \tag{18}$$

where

$$k(\tau) = \delta/ml(\tau),\ \rho(\tau) = 2l'(\tau)/l(\tau),\ v^2(\tau) = g/l(\tau),\ \varepsilon_1(\tau) = g/6l(\tau) = \tfrac{1}{6}v^2(\tau),\ l' = dl(\tau)/d\tau,\ \tau = \varepsilon t.$$

Here $n = 2,\ j = 1, 2$, eigenvalues are $\lambda_1(\tau)$ and $\lambda_2(\tau)$; relations between eigenvalues and coefficients $\lambda_1 + \lambda_2 = -2k$, $\lambda_1\lambda_2 = v^2$ or two eigenvalues are $\lambda_{1,2} = -k \pm \omega, \omega^2 = k^2 - v^2$. In this case $x_0 = a_1 + a_2$, $f^{(0)} = -\frac{1}{2}\rho(\lambda_1 - \lambda_2)(a_1 - a_2) + \varepsilon^{-1}\varepsilon_1(a_1^3 + \varepsilon_1 3a_1^2a_2 + 3a_1a_2^2 + a_2^3)$ and $\Omega \equiv \lambda_1 a_1(\partial/\partial a_1) + \lambda_2 a_2(\partial/\partial a_2)$. Now substituting the values of n, j and $f^{(0)}$ in the general equation (5) and following the assumption (discussed in Section 2), we obtain Eqs. (8)–(10) as

$$(\Omega - \lambda_1)(\Omega - \lambda_2)u_1 = \varepsilon^{-1}\varepsilon_1(a_1^3 + a_2^3), \tag{19}$$

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 = -\tfrac{1}{2}\rho(\lambda_1 - \lambda_2)a_1 + 3\varepsilon^{-1}\varepsilon_1 a_1^2 a_2, \tag{20}$$

and

$$(\Omega - \lambda_1)A_2 + \lambda_2' a_2 = -\tfrac{1}{2}\rho(\lambda_1 - \lambda_2)a_2 + 3\varepsilon^{-1}\varepsilon_1 a_1 a_2^2. \tag{21}$$

We have discussed in Section 2 that the particular solutions of Eqs. (19)–(21) would be found by simply replacing Ω by $m_1\lambda_1 + \lambda_2 m_2$. To illustrate this matter, let us consider the second term with $a_1^2 a_2$ on the right-hand side of Eq. (20). Here, we only replace Ω by $2\lambda_1 + \lambda_2$ or $\Omega - \lambda_2$ by $2\lambda_1$. Similarly, for the first term with a_1 (included left sided term $\lambda_1' a_1$), Ω should be replaced by λ_1 or $\Omega - \lambda_2$ by $\lambda_1 - \lambda_2$. Therefore, the particular solutions of Eq. (20) is

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \tfrac{1}{2}\rho a_1 + \frac{3\varepsilon^{-1}\varepsilon_1 a_1^2 a_2}{2\lambda_1}. \tag{22}$$

In a similar procedure, we can obtain the particular solutions of Eqs. (21) and (19) as

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \tfrac{1}{2}\rho a_2 + \frac{3\varepsilon^{-1}\varepsilon_1 a_1 a_2^2}{2\lambda_2}, \tag{23}$$

and

$$u_1 = \varepsilon^{-1}\varepsilon_1\left(\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} + \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)}\right). \tag{24}$$

Substituting the values of A_1 and A_2 from Eqs. (22) and (23) into Eq. (4), we obtain

$$\begin{aligned} \dot a_1 &= -\lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_1} - \frac{l' a_1}{l}\right) + \frac{3\varepsilon_1 a_1^2 a_2}{2\lambda_1}, \\ \dot a_2 &= - - \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_1} - \frac{l' a_2}{l}\right) + \frac{3\varepsilon_1 a_1 a_2^2}{2\lambda_2}. \end{aligned} \tag{25}$$

Now we have to solve Eq. (25) for a_1 and a_2; but it is hard to say that Eq. (25) has an exact solution or not. Most of the cases (i.e., under-damped or over-damped or critically damped), we are unable to find an exact solution of Eq. (4) when a non-linear system posses strong linear damping force(s) [6,7,10,13,15–17]. In the case of an over-damped system, Murty and Deekshatulu [16] replace the terms with small parameter ε, by their respective unperturbed value (i.e., $a_j(t)$ by $a_{j,0}e^{\lambda_j(\tau_0)t}$), since x together with all $a_j(t)$ die out quickly. Within this time interval, the difference between $a_j(t)$ and $a_{j,0}e^{\lambda_j(\tau_0)t}$ occurs in an order of ε only. But, in the case of oscillations with small damping or without damping, this is definitely wrong. In this case, x and $a_j(t)$ die out slowly and the difference of $a_j(t)$ and $a_{j,0}e^{\lambda_j(\tau_0)t}$ increases with t and after a long time the difference occurs by more than an order of ε. However, for the system with strong damping effects, this assumption is also correct (see Ref. [17] for details). Thus in the case of an over-damped system, we obtain an approximate solution of (25) as

$$\begin{aligned} a_1 &\cong \frac{a_1^0 l_0}{l}\exp\left(-\int_0^t \lambda_1 \mathrm{d}t - \int_0^{\lambda_1(\tau)} \frac{\mathrm{d}\lambda_1}{\lambda_1 - \lambda_1}\right) + \frac{3\varepsilon_1 a_{1,0} a_2 (e^{(\lambda_1 + \lambda_2)} - 1)}{2\lambda_1(\lambda_1 + \lambda_2)}, \\ a_2 &\cong \frac{a_2^0 l_0}{l}\exp\left(-\int_0^t \lambda_2 \mathrm{d}t + \int_0^{\lambda_2(\tau)} \frac{\mathrm{d}\lambda_2}{\lambda_1 - \lambda_1}\right) + \frac{3\varepsilon_1 a_{1,0} a_2 (e^{(\lambda_1 + \lambda_2)} - 1)}{2\lambda_2(\lambda_1 + \lambda_2)} \end{aligned} \tag{26}$$

Therefore, we obtain a first order solution of Eq. (18) for all real values of λ_1 and λ_2

$$x(t, \varepsilon) = a_1 + a_2 + \varepsilon u_1, \tag{27}$$

where a_1 and a_2 are given by Eq. (26) and u_1 by Eq. (24).

We can find Bojadziev and Edwards' over-damped solutions from Eq. (27) together with Eqs. (25) and (24) under the substitution $a_1 = \frac{1}{2}a\mathrm{e}^{\varphi}$ and $a_2 = \pm\frac{1}{2}a\mathrm{e}^{-\varphi}$ (a and φ are known as amplitude and phase), which transform (25) to

$$\dot{a} = -ka + \varepsilon\left(-\frac{\omega' a}{2\omega} - \frac{l' a}{l} - \frac{3ka^3}{8\nu^2}\right),$$
$$\dot{\varphi} = \omega + \varepsilon\left(\frac{k' a}{2\omega} - \frac{3\omega a^2}{8\nu^2}\right). \tag{28}$$

It is clear that the variational equation (28) is an exact form of KBM solution (see also Appendix A). Now, substituting the values of k, ρ, ν and ε_1 into Eq. (28), we obtain

$$\dot{a} = -\frac{\delta a}{ml} + \varepsilon\left(\frac{\delta^2 l'/4m^2l^2}{\delta^2/ml - g} - \frac{3l'}{4l}\right)a - \frac{\delta a^3}{16ml},$$
$$\dot{\varphi} = \sqrt{\frac{\delta^2}{m^2l^2} - \frac{g}{l}} \times \left(1 - \frac{\varepsilon\delta l'/ml^2}{2\left(\delta^2/m^2l^2 - g/l\right)} - \frac{a^2}{16}\right). \tag{29}$$

We are able to determine an appgroximate solution of Eq. (29) when l changes linearly with time, i.e., $l = l_0 + \varepsilon l_1 t$. Substituting this value of l, $l^{-1} \cong l_0^{-1} - \varepsilon l_1 l_0^{-2} t$, into Eq. (29) and simplifying, we obtain an approximate solution of Eq. (29) (in order of ε [7]) as

$$a = \frac{a_0 \exp\left(-\delta t/ml_0 + \varepsilon\left(\frac{\delta^2 l_1/4m^2l_0^2}{\delta^2/ml_0 - g} - 3l_1/4l\right)t + \delta\varepsilon l_1 t^2/2ml_0^2\right)}{\sqrt{1 + \frac{1}{16}a_0^2\left(1 - \mathrm{e}^{-2\delta t/ml_0}\right)}},$$
$$\varphi = \varphi_0 + \left(\sqrt{\frac{\delta^2}{m^2l_0^2} - \frac{g}{l_0}}\left(t - \frac{\int_0^t a^2(t)\,\mathrm{d}t}{16}\right) - \frac{\varepsilon\delta l_1 t/ml_0^2}{2\sqrt{\delta^2/m^2l_0^2 - g/l_0}}\right) + \frac{\varepsilon l_1 t^2\left(\delta^2/m^2l_0^2 - g/2l_0\right)}{\delta^2/m^2l_0^2 - g/l_0}. \tag{30}$$

This result is similar to that obtained by Bojadziev and Edwards [7], but not identical. Bojadziev and Edwards considered equation $ml^2(\tau)\ddot{\theta} + 2\delta l(\tau)\dot{\theta} + mgl(\tau)\sin\theta = 0$ instead of Eq. (17) [7]. However, actual pendulum equation with varying length is Eq. (17) [18]. The above solution is valid even when the length of the pendulum is considered constant. Moreover, this solution is used as an under-damped. First, consider the case of constant length, i.e., $l_1 = 0$. We can take a limit $l_1 \to 0$ and then Eq. (30) becomes

$$a = \frac{a_0 \mathrm{e}^{-\delta t/ml_0}}{\sqrt{1 + \frac{1}{16}a_0^2\left(1 - \mathrm{e}^{-2\delta t/ml_0}\right)}},$$
$$\varphi = \varphi_0 + \sqrt{\frac{\delta^2}{m^2l_0^2} - \frac{g}{l_0}}\left(t + \frac{m}{2\delta}\ln\left(1 + \frac{1}{16}a_0^2\left(1 - \mathrm{e}^{-2\delta t/ml_0}\right)\right)\right). \tag{31}$$

Solution (31) is identical to Murty's solution [5], who first presented the unified formula for solving second order over-damped and under-damped systems. However, for an under-damped system, we know the inequality, $\delta^2/m^2l_0^2 < g/l_0$. Therefore, we should replace φ and φ_0 respectively by $i\varphi$ and $i\varphi_0$ to obtain the amplitude and phase (in real form) of an under-damped solution.

Thus under the transformations $a_1 = \frac{1}{2}e^{\varphi}, a_2 = \frac{1}{2}e^{\pm\varphi}$, we obtain two solutions of Eq. (18):

$$x = a\cosh\varphi + \frac{\varepsilon_1 a^3\left((k^2+2\omega^2)\cosh 3\varphi + 3k\omega\sinh 3\varphi\right)}{16v^2(k^2-4\omega^2)}, \tag{32}$$

and

$$x = a\sinh\varphi + \frac{\varepsilon_1 a^3\left(3k\omega\cosh 3\varphi + (k^2+2\omega^2)\sinh 3\varphi\right)}{16v^2(k^2-4\omega^2)}, \tag{33}$$

where a and φ (whether real or imaginary) are given by Eq. (30), and k, ω, ν etc. are defined above. It is obvious that both solutions represent an under-damped system when φ and ω are imaginary (however, they have only a phase difference). Therefore, we may use one solution arbitrarily for different initial conditions. On the other hand, we have to use one of these solutions (either Eq. (32) or Eq. (33)) for a given set of initial conditions, when the system is an over-damped (see Ref. [5] or Ref. [7] or Ref. [12] for details). Thus it is better to use Eq. (27) as an over-damped solution instead of Eq. (32) or Eq. (33), since we can use Eq. (27) arbitrarily for different initial conditions.

Now we consider a special case of an under-damped system when we may neglect the terms with δ^2 and $\varepsilon\delta$ of Eq. (29), so that we have

$$\dot a = -\frac{\delta a}{ml} - \frac{3\varepsilon l' a}{4l} - \frac{\delta a^3}{16ml}, \quad \dot\varphi = \sqrt{-\frac{g}{l}} \times \left(1 - \frac{a^2}{16}\right). \tag{34}$$

The solution of Eq. (34) is

$$a(t) = \frac{a_0\left(l_0/l\right)^{\delta/\varepsilon m l_1 + 3/4}}{\sqrt{1+\Phi(t)}}, \quad \Phi(t) = \frac{\delta\left(l_0^{-(2\delta/\varepsilon m l_1)-(3/2)} - l^{-(2\delta/\varepsilon m l_1)-(3/2)}\right)}{16\delta + 12m\varepsilon l_1},$$
$$\varphi = \varphi_0 + \int_0^t \sqrt{-\frac{g}{l}} \times \left(1 - \frac{a^2(t)}{16}\right) dt. \tag{35}$$

It is obvious that the change of $\Phi(t)$ is small. We neglect this term when $\delta = O(\varepsilon)$. Thus for a small damping effect, Eq. (35) may be written in the simple form

$$a(t) = a_0\left(\frac{l_0}{l}\right)^{(\delta/\varepsilon m l_1)+(3/4)}, \varphi = \varphi_0 + i\int_0^t \sqrt{\frac{g}{l}} \times \left(1 - \frac{a^2(t)}{16}\right) dt. \tag{36}$$

Therefore, an under-damped solution of a pendulum, whose length is changing linearly with time, is

$$x(t,\varepsilon) = a(t)\cos\psi - \frac{1}{192}a^3(t)\cos 3\psi, \ \psi = \psi_0 + \int_0^t \sqrt{\frac{g}{l}} \times \left(1 - \frac{a^2(t)}{16}\right) dt, \tag{37}$$

where a is given by Eq. (36). This solution is similar to that obtained in Ref. [3] (see also Ref. [18]). If l is constant, Eq. (37) reduces to $x(t,\varepsilon) = a_0\ cos\psi - \frac{1}{192}a_0^3\cos 3\psi,\ \psi = \psi_0 + \sqrt{gl_0^{-1}}(1 - \frac{1}{16}a_0^2)$. This solution is identical to that obtained by Bogoliubov and Mitroposkii [2] (see also Ref. [5]).

3.2. Third-order nonlinear systems

Let us consider a third order non-linear differential equation

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon\, x^3. \tag{38}$$

Here $n = 3$, $j = 1, 2, 3$ and eigenvalues are λ_1, λ_2 and λ_2; $x_0 = a_1 + a_2 + a_3$ and the function $f^{(0)} = a_1^3 + 3a_1^2a_2 + 3a_1^2a_3 + 3a_1a_2^2 + 3a_1a_3^2 + 6a_1a_2a_3 + a_2^3 + 3a_2^2a_3 + 3a_2a_3^2 + a_3^3$. Substituting the values of n, j and $f^{(0)}$ in Eq. (5) and following assumptions discussed in Section 2 (for an odd value of n) and similar steps as presented in Section 3.1, we obtain four equations for A_1, A_2, A_3 and u_1 whose solutions are, respectively,

$$\begin{aligned}
A_1 &= -\frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{a_1^3}{(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)} + \frac{6a_1a_2a_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)},\\
A_2 &= -\frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{3a_1^2a_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2 - \lambda_3)} + \frac{3a_2^2a_3}{2\lambda_2(2\lambda_2 + \lambda_3 - \lambda_1)},\\
A_3 &= -\frac{(2\lambda_3 - \lambda_1 - \lambda_2)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{3a_1^2a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 - \lambda_2 + \lambda_3)} + \frac{3a_2a_3^2}{2\lambda_3(2\lambda_3 + \lambda_2 - \lambda_1)},
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
u_1 = {}& \frac{3a_1a_2^2}{2\lambda_2(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)} + \frac{3a_1a_3^2}{2\lambda_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2 + 2\lambda_3)}\\
&+ \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)} + \frac{a_3^3}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)}.
\end{aligned} \tag{40}$$

Substituting the values of A_1, A_2 and A_3 from Eq. (39) into Eq. (4), we obtain

$$\begin{aligned}
\dot{a}_1 &= -\lambda_1 a_1 + \varepsilon\left(-\frac{(2\lambda_1 - \lambda_2 - \lambda_3)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{a_1^3}{(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)} + \frac{6a_1a_2a_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)}\right),\\
\dot{a}_2 &= -\lambda_2 a_2 + \varepsilon\left(-\frac{(2\lambda_2 - \lambda_1 - \lambda_3)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{3a_1^2a_2}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2 - \lambda_3)} + \frac{3a_2^2a_3}{2\lambda_2(2\lambda_2 + \lambda_3 - \lambda_1)}\right),\\
\dot{a}_3 &= -\lambda_3 a_3 + \varepsilon\left(-\frac{(2\lambda_3 - \lambda_1 - \lambda_2)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{3a_1^2a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 - \lambda_2 + \lambda_3)} + \frac{3a_2a_3^2}{2\lambda_3(2\lambda_3 + \lambda_2 - \lambda_1)}\right).
\end{aligned} \tag{41}$$

For a damped solution of Eq. (38), we may substitute $\lambda_1(\tau) = -\lambda(\tau)$, $\lambda_{1,2}(\tau) = -\mu(\tau) \pm i\omega(\tau)$ and $a_1 = a$, $a_2 = \frac{1}{2}be^{i\varphi}$, $a_3 = \frac{1}{2}be^{-i\varphi}$ into Eqs. (41) and (40) and then simplifying them, we obtain

$$\begin{aligned}
\dot{a} &= -\lambda(\tau)a + \varepsilon(l_0 a + l_1 a^3 + l_2 ab^2),\\
\dot{b} &= -\mu(\tau)b + \varepsilon(m_0 b + m_1 a^2 b + m_2 b^3),\\
\dot{\varphi} &= \omega(\tau) + \varepsilon(n_0 + n_1 a^2 + n_2 b^2).
\end{aligned} \tag{42}$$

and

$$u_1 = ab^2(c_2\cos 2\varphi + d_2\sin 2\varphi) + b^3(c_3\cos 3\varphi + d_3\sin 3\varphi), \tag{43}$$

where

$$\begin{aligned}
l_0 &= \frac{2(\lambda-\mu)\lambda'}{(\lambda-\mu)^2+\omega^2}, l_1 = \frac{1}{(3\lambda-\mu)^2+\omega^2}, \;\; l_2 = \frac{3}{2((\lambda+\mu)^2+\omega^2)},\\
m_0 &= \frac{2(\lambda-\mu)\mu'\omega - ((\lambda-\mu)^2+3\omega^2)\omega'}{2\omega((\lambda-\mu)^2+\omega^2)}, \;\; n_0 = -\frac{((\lambda-\mu)^2+3\omega^2)\mu' + 2(\lambda-\mu)\omega\omega'}{2\omega((\lambda-\mu)^2+\omega^2)},\\
m_1 &= \frac{3(\lambda^2+\lambda\mu-\omega^2)}{2(\lambda^2+\omega^2)\,((\lambda+\mu)^2+\omega^2)}, \;\; n_1 = \frac{3(2\lambda+\mu)\omega}{2(\lambda^2+\omega^2)\,((\lambda+\mu)^2+\omega^2)},\\
m_2 &= \frac{3\omega(\mu(-\lambda+3\,\mu)-\omega^2)}{8(\mu^2+\omega^2)\,((\lambda-3\mu)^2+\omega^2)}, \;\; n_2 = \frac{3\omega(-\lambda+4\,\mu)}{8(\mu^2+\omega^2)\,((\lambda-3\mu)^2+\omega^2)},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
c_2 &= \frac{3(-\mu(\lambda+\mu)^2+(4\lambda+7\mu)\omega^2)}{4(\mu^2+\omega^2)\,((\lambda+\mu)^2+\omega^2)\,((\lambda+\mu)^2+9\omega^2)},\\
d_2 &= \frac{3\omega((\lambda+\mu)(\lambda+5\mu)-3\omega^2)}{4(\mu^2+\omega^2)\,((\lambda+\mu)^2+\omega^2)\,((\lambda+\mu)^2+9\omega^2)},\\
c_3 &= \frac{\mu^2(\lambda-3\mu)+(-2\lambda+15\mu)\omega^2}{16(\mu^2+\omega^2)\,(\mu^2+4\omega^2)\,((\lambda-3\mu)^2+9\omega^2)},\\
d_3 &= \frac{-3\omega(\mu(\lambda-3\mu)+2\omega^2)}{16(\mu^2+\omega^2)\,(\mu^2+4\omega^2)\,((\lambda-3\mu)^2+9\omega^2)}.
\end{aligned} \tag{45}$$

Thus the first order solution of Eq. (38) is

$$x(t,\varepsilon) = a + b\cos\varphi + \varepsilon u_1, \tag{46}$$

where a, b and φ are solution of Eq. (42) and u_1 is given by Eq. (43). In general, Eqs. (42) are solved by a numerical procedure [10,12,13]. In this case, the perturbation method facilitates only the numerical method. The variables a, b and φ change slowly with time. So, it requires the numerical calculation of a few number of points. Contrary, a direct attempt to solve the Eq. (38) dealing with harmonic terms in solution (46), namely $b\cos\varphi$, requires the numerical calculation of a great number of points. Often one is not interested in only the oscillating processes itself, i.e., finding the x in terms of t, but mainly in the behavior of the amplitudes a, b and the phase φ, which as t increases characterize the oscillating processes [10].

4. General discussion of the results

The core of this study is to find a simple and unified method for solving nth order ordinary differential equations with slowly varying coefficients. The method was originally developed for obtaining the periodic solutions of second order non-linear systems by Kryolv and Bogoliubov [1] and later amplified and justified by Bogoliubov and Mitroploskii [2]. Then the method was extended to a strong damped oscillatory system by Popov [4]. Murty et al. [19] used Popov's method to obtain an over-damped solution of a second order non-linear differential equation and that was the basis of unified theory of Murty [5]. Most probably, Osiniskii [9] first studied a third order non-linear mechanical elastic system with internal friction and relaxation (by extended KBM method). However, Bojadziev [7] modified his solution. Mulholand [8] found only oscillatory part of a third order non-linear differential equation, $\dddot{x} + \ddot{x} + \dot{x} + x = \varepsilon(1 - x^2 - \dot{x}^2 - \ddot{x}^2)(\ddot{x} + \dot{x})$, by Kryolv and Bogoliubov,s method [1]. It is noted that most of the cases, first order solution (yet not improved version, i.e., u_1 is not considered in the solution) was found for the third order systems, since it was a laborious task to determine u_1. However, Pavilidis extended this method to an n-dimensional biological system. Then many authors (e.g., Bojadziev [20,21], Bojadziev and Chan [22], Dutt, Ghosh and Karmaker [23], Lin and Khan [24]) studied some biological, biomedical and biochemical systems. From the above references it is clear that KBM method is being used not only in mechanical system or electric circuit theory, but also in a wide variety of several branches of sciences and engineering. Nowadays, the method is not limited to second order problems, but also useful in third order or more than third order nonlinear systems.

The method (concerned with this paper) is not an exact form of unified KBM method. The new form has been chosen to remove some difficulties of the formal method. We have already discussed in Section 3.1 that two solutions are needed for a second order over-damped system (depending on a given set of initial conditions, i.e., $a_1 = \frac{1}{2}ae^{\varphi}, a_2 = \frac{1}{2}e^{-\varphi}$ is chosen when both $a_{1,0}$ and $a_{2,0}$ are positive while $a_1 = \frac{1}{2}ae^{\varphi}, a_2 = -\frac{1}{2}e^{-\varphi}$ is chosen when $a_{1,0}$ is positive and $a_{2,0}$ is negative) in accordance with the standard form of unified KBM method (see Appendix A). In this case, our solution (27) can be used arbitrarily for different initial conditions. Similarly, two solutions are needed for a third order over-damped system (see Ref. [12] for details). More difficulties arise when a fourth order or fifth order system poses all real roots. In these cases four different solutions are needed for different set of initial conditions. Thus, the standard form of unified method is a cumbersome procedure to solve initial value problems (over-damped).

There is another demerit of the present form of extended KBM method. We have to solve two simultaneous differential equations for amplitude and phase, and a partial differential equation of u_1 involving two independent variables, amplitude and phase (see Appendix A). Increasing with the order of differential equations a set of simultaneous equations appeared. In these cases our new method is easier than the former. On the other hand, we are able to solve all the equations of $A_j, j = 1, 2, \ldots, n$ including u_1 by a unified and simple formula. Moreover, for an even or odd value of n, the procedure is also unified. The disadvantage of this method is that we should transform the solution by a substitution formula when the system is under-damped. However, this method is yet easier than extended form (by Popov [4]) of KBM method.

5. Conclusion

A general formula is presented by the unified KBM method [1–5] for obtaining the transients response of non-linear systems governed by an nth order ordinary differential system with a small non-linearity. The solutions for oscillatory, damped oscillatory and non-oscillatory cases can be derived from a single equation (5). Thus there is no longer any need to treat three cases separately.

Moreover, the method is also independent of the order of the differential equation. The solution reduces to that obtained by Shamsul Alam [15] previous method when the coefficients become constant.

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Appendix A. Method of Bojadziev and Edwards [7] and other solutions obtained in Refs.[2]–[5]

When $n = 2$, Eqs. (8)–(10) of Section 2 take the forms

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = \sum_{m_1=0, m_1=0}^{\infty,\infty} F_{m_1,m_2} a^{m_1} a^{m_2}, \tag{A.1}$$

where $m_1 - m_2 = \pm 1$,

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_1 + \lambda_1' a_1 = \sum_{m_1=0, m_1=0}^{\infty,\infty} F_{m_1,m_2} a^{m_1} a^{m_2},\ m_1 - m_2 = 1, \tag{A.2}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_2 + \lambda_1' a_1 = \sum_{m_1=0, m_1=0}^{\infty,\infty} F_{m_1,m_2} a^{m_1} a^{m_2},\ m_1 - m_2 = -1. \tag{A.3}$$

We have already mentioned that our solution can be brought to formal KBM solution under a transformation:

$$a_1 = \tfrac{1}{2} a e^{\varphi}, a_2 = \tfrac{1}{2} e^{-\varphi}. \tag{A.4}$$

For Eq. (A.4), we obtain the following results

$$a_1 \frac{\partial}{\partial a_1} = \frac{1}{2}\left(a\frac{\partial}{\partial a} + \frac{\partial}{\partial \varphi}\right),\ a_2 \frac{\partial}{\partial a_2} = \frac{1}{2}\left(a\frac{\partial}{\partial a} - \frac{\partial}{\partial \varphi}\right),$$

or

$$\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_1 a_1 \frac{\partial}{\partial a_1} = \frac{1}{2}\left((\lambda_1 + \lambda_2) a \frac{\partial}{\partial a} + (\lambda_1 - \lambda_2)\frac{\partial}{\partial \varphi}\right). \tag{A.5}$$

Substituting $\lambda_1 = -k(\tau) + \omega(\tau)$, $\lambda_2 = -k(\tau) - \omega(\tau)$ into Eq. (A.5), we obtain

$$\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_1 a_1 \frac{\partial}{\partial a_1} = -ka\frac{\partial}{\partial a} + \omega \frac{\partial}{\partial \varphi}. \tag{A.6}$$

Differentiating a_1 and a_2 with respect to t and utilizing relations (4), we obtain $\dot{a}_1 = \lambda_1 a_1 + \varepsilon A_1 + \varepsilon^2 \cdots = \frac{1}{2}(\dot{a} + a\dot{\varphi})e^{\varphi}, \dot{a}_2 = \lambda_2 a_2 + \varepsilon A_2 + \varepsilon^2 \cdots = \frac{1}{2}(\dot{a} - a\dot{\varphi})e^{-\varphi}$, or,

$$\begin{aligned} \tfrac{1}{2}(-k + \omega) a e^{\varphi} + \varepsilon A_1 + \varepsilon^2 \cdots &= \tfrac{1}{2}(\dot{a} + a\dot{\varphi}) e^{\varphi}, \\ \tfrac{1}{2}(-k - \omega) a e^{-\varphi} + \varepsilon A_2 + \varepsilon^2 \cdots &= \tfrac{1}{2}(\dot{a} - a\dot{\varphi}) e^{-\varphi}. \end{aligned} \tag{A.7}$$

With the help of Eq. (A.6), we can rewrite equations Eqs. (A.2) and (A.3) as

$$\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k-\omega\right)A_1+\tfrac{1}{2}(-k'+\omega')a\mathrm{e}^{\varphi} = \sum_{m_1=0,m_1=0}^{\infty,\infty} F_{m_1,m_2}(\tfrac{1}{2}a)^{(m_1+m_2)}\mathrm{e}^{\varphi}, m_1-m_2=1, \tag{A.8}$$

$$\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k+\omega\right)A_2+\tfrac{1}{2}(-k'-\omega')a\mathrm{e}^{-\varphi} = \sum_{m_1=0,m_1=0}^{\infty,\infty} F_{m_1,m_2}(\tfrac{1}{2}a)^{(m_1+m_2)}\mathrm{e}^{-\varphi}, m_1-m_2=-1, \tag{A.9}$$

or,

$$\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k-\omega\right)A_1+\tfrac{1}{2}(-k'+\omega')a\mathrm{e}^{\varphi}=\sum_{r=1}^{\infty}F_{r,r-1}(\tfrac{1}{2}a)^{2r-1}\mathrm{e}^{\varphi}, \tag{A.10}$$

$$\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k+\omega\right)A_2+\tfrac{1}{2}(-k'-\omega')a\mathrm{e}^{-\varphi}=\sum_{r=1}^{\infty}F_{r,r-1}(\tfrac{1}{2}a)^{2r-1}\mathrm{e}^{-\varphi}. \tag{A.11}$$

Multiplying Eqs. (A.10) and (A.11) respectively, by $\mathrm{e}^{-\varphi}$ and e^{φ}, and then rearranging them, we obtain

$$2\omega A_1\mathrm{e}^{-\varphi}+\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k\right)(A_1\mathrm{e}^{-\varphi})+\tfrac{1}{2}(-k'+\omega')a=\sum_{r=1}^{\infty}F_{r,r-1}(\tfrac{1}{2}a)^{2r-1}, \tag{A.12}$$

$$-2\omega A_2\mathrm{e}^{\varphi}+\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k\right)(A_2\mathrm{e}^{\varphi})+\tfrac{1}{2}(-k'-\omega')a=\sum_{r=1}^{\infty}F_{r-1,r}(\tfrac{1}{2}a)^{2r-1} \tag{A.13}$$

Adding and subtracting Eqs. (A.12) and (A.13), we obtain

$$2\omega(A_1e^{-\varphi}-A_2\mathrm{e}^{\varphi})+\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k\right)(A_1e^{-\varphi}+A_2\mathrm{e}^{\varphi})-k'a = \sum_{r=1}^{\infty}(F_{r,r-1}+F_{r-1,r})(\tfrac{1}{2}a)^{2r-1}, \tag{A.14}$$

$$2\omega(A_1\mathrm{e}^{-\varphi}+A_2\mathrm{e}^{\varphi})+\left(-ka\frac{\partial}{\partial a}+\omega\frac{\partial}{\partial\varphi}+k\right)(A_1\mathrm{e}^{-\varphi}-A_2e^{\varphi})+\omega'a = \sum_{r=1}^{\infty}(F_{r,r-1}-F_{r-1,r})(\tfrac{1}{2}a)^{2r-1}. \tag{A.15}$$

Again, multiplying two equations of Eq. (A.7), respectively, by $\mathrm{e}^{-\varphi}$ and e^{φ}, and then adding and subtracting, we obtain

$$\begin{aligned}\dot a &= -ka+\varepsilon(A_1\mathrm{e}^{-\varphi}+A_2\mathrm{e}^{\varphi})+\varepsilon^2\cdots,\\ a\dot\varphi &= a\omega+\varepsilon(A_1\mathrm{e}^{-\varphi}-A_2\mathrm{e}^{\varphi})+\varepsilon^2\cdots.\end{aligned} \tag{A.16}$$

It is obvious that the particular solutions of Eqs. (A.14) and (A.15) are independent of φ, since the right-hand sides of these equations contain a only; so that these particular solutions can be found from more simple equations:

$$2\omega(A_1 e^{-\varphi} - A_2 e^{\varphi}) + \left(-ka\frac{d}{da} + k\right)(A_1 e^{-\varphi} + A_2 e^{\varphi}) - k'a = \sum_{r=1}^{\infty}(F_{r,r-1} + F_{r-1,r})\left(\tfrac{1}{2}a\right)^{2r-1}, \tag{A.17}$$

$$2\omega(A_1 e^{-\varphi} + A_2 e^{\varphi}) + \left(-ka\frac{d}{da} + k\right)(A_1 e^{-\varphi} - A_2 e^{\varphi}) + \omega'a = \sum_{r=1}^{\infty}(F_{r,r-1} - F_{r-1,r})\left(\tfrac{1}{2}a\right)^{2r-1}. \tag{A.18}$$

Now if we replace $A_1 e^{-\varphi} - A_2 e^{\varphi} = \tilde{A}_1(a)$ and $A_1 e^{-\varphi} - A_2 e^{\varphi} = a\tilde{B}_1(a)$ (where $\tilde{A}_1$ and $\tilde{B}_1$ are usual notations) Eqs. (A.13)–(A.15) reduce to

$$\begin{aligned} \dot{a} &= -ka + \varepsilon\tilde{A}_1(a) + \varepsilon^2\cdots, \\ \dot{\varphi} &= a + \varepsilon\tilde{B}_1(a) + \varepsilon^2\cdots, \end{aligned} \tag{A.19}$$

$$2\omega\tilde{B}_1 + \left(-ka\frac{d}{da} + k\right)\tilde{A}_1 - k'a = \sum_{r=1}^{\infty}(F_{r,r-1} + F_{r-1,r})\left(\tfrac{1}{2}a\right)^{2r-1}, \tag{A.20}$$

and

$$2\omega\tilde{A}_1 + \left(-ka\frac{d}{da} - \omega\right)a\tilde{B}_1 + \omega'a = \sum_{r=1}^{\infty}(F_{r,r-1} - F_{r-1,r})\left(\tfrac{1}{2}a\right)^{2r-1},$$

or

$$2\omega\tilde{A}_1 - ka^2\frac{d\tilde{B}_1}{da} + \omega'a = \sum_{r=1}^{\infty}(F_{r,r-1} - F_{r-1,r})\left(\tfrac{1}{2}a\right)^{2r-1}. \tag{A.21}$$

Eqs. (A.20)–(A.21) can be rewritten as

$$-\frac{k'a}{2} + \frac{\omega'a}{2} + \frac{1}{2}\left(-ka\frac{d\tilde{A}_1}{da} + k\tilde{A}_1 + 2\omega a\tilde{B}_1\right) + \frac{1}{2}\left(2\omega\tilde{A}_1 - ka^2\frac{d\tilde{B}_1}{da}\right) = \sum_{r=1}^{\infty}F_{r-1,r}\left(\tfrac{1}{2}a\right)^{2r-1} \tag{A.22}$$

$$-\frac{k'a}{2} - \frac{\omega'a}{2} + \frac{1}{2}\left(-ka\frac{d\tilde{A}_1}{da} + k\tilde{A}_1 + 2\omega a\tilde{B}_1\right) - \frac{1}{2}\left(2\omega\tilde{A}_1 - ka^2\frac{d\tilde{B}_1}{da}\right) = \sum_{r=1}^{\infty}F_{r-1,r}\left(\tfrac{1}{2}a\right)^{2r-1}. \tag{A.23}$$

With the help of Eq. (A.6), Eq. (A.1) can be written as

$$\left(-ka\frac{\partial}{\partial a} + \frac{\partial}{\partial\varphi} - k + \omega\right)\left(-ka\frac{\partial}{\partial a} + \frac{\partial}{\partial\varphi} - k - \omega\right)u_1 = \sum_{m_1,m_2=0,m_1-m_2\neq\pm1}^{\infty}F_{m_1,m_2}\left(\tfrac{1}{2}a\right)^{m_1+m_2}e^{(m_1-m_2)\varphi}. \tag{A.24}$$

or

$$\left(-ka\frac{\partial}{\partial a} + \frac{\partial}{\partial\varphi}\right)^2 u_1 + 2c_1\left(-ka\frac{\partial}{\partial a} + \frac{\partial}{\partial\varphi}\right)u_1 + c_2u_1 = \sum_{m_1,m_2=0,m_1-m_2\neq\pm1}^{\infty}F_{m_1,m_2}\left(\tfrac{1}{2}a\right)^{m_1+m_2}e^{(m_1-m_2)\varphi}. \tag{A.25}$$

Eqs. (A.22)–(A.23) and (A.25) are exact forms of those obtained by Bojadziev and Edwards [7]. If the coefficients are constants i.e., $k' = \omega' = 0$, these equations are similar to those obtained by Murty [5]. Equations similar to (A.22),(A.23) and (A.25) would be found if we use transformations $a_1 = \frac{1}{2}a\mathrm{e}^{\varphi}, a_2 = -\frac{1}{2}\mathrm{e}^{-\varphi}$. If we replace ω by $i\omega$, $\tilde{B}_1$ by $i\tilde{B}_1$ and substitute $k' = \omega' = 0$ into Eqs. (A.22)–(A.23) and (A.25), they transform to those obtained by Popov [4] while they transform to those obtained in Ref. [2] when $k = k' = \omega' = 0$. Again these will be transformed to those obtained in Ref. [3] if $k = O(\varepsilon)$ and $\omega = \omega(\tau)$. Similar verification can be made when $n \geqslant 3$.

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